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Fixed-point-free embeddings of digraphs with small size

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Abstract

We prove that every digraph of order n and size at most $\frac{7}{4}n - 81$ is embeddable in its complement. Moreover, for such digraphs there are embeddings without fixed points.

Keywords: Embedding of digraphs; Packing of digraphs

1. Introduction

We consider only finite graphs and digraphs without loops and multiple edges and arcs. Our terminology and notation are standard unless otherwise stated.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We say that G is *embeddable* (in its complement) if there is a permutation σ on $V(G)$ such that if an edge xy belongs to $E(G)$ then $\sigma(x)\sigma(y)$ does not belong to $E(G)$. The permutation σ is called an *embedding* of the graph G.

Similarly, let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. We say that D is *embeddable* (in its complement) if there is a permutation σ on $V(D)$ such that $(\sigma(x), \sigma(y)) \notin A(D)$ for every arc $(x, y) \in A(D)$. The permutation σ is called an *embedding* of the digraph D. A digraph D is *self-complementary* if it is isomorphic to its complement.

The problem of finding the maximum number $f(n)$ such that every graph of order n with size at most $f(n)$ is embeddable was independently solved in [3,9,2].

Theorem 1. *Let G be a graph of order n. If $|E(G)| \leqslant n - 2$ then G is embeddable.*

The example of the star of order n shows that Theorem 1 cannot be improved by raising the size of G. However, the above result has been improved in many ways.

All non-embeddable graphs of order n and size $n - 1$ are given in [4] and all non-embeddable graphs of order and size n are characterized in [6].

In this paper we deal with some additional properties of embeddings. Some improvement of Theorem 1 related to the structure of embeddings is proved in [10].

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Theorem 2. *Let G be a graph of order n. If $|E(G)| \leqslant n-2$ then there is an embedding σ of G without fixed points i.e. $\sigma(x) \neq x$ for $x \in V(G)$.*

The above theorem cannot be improved by increasing the number of edges. Let us consider graphs $K_{1,2} \cup K_3$ and $K_{1,3} \cup K_3$. Observe that these graphs are embeddable. It is not difficult to check that every embedding of $K_{1,2} \cup K_3$ has to map the centre of the star $K_{1,2}$ onto itself. Similarly, every embedding of $K_{1,3} \cup K_3$ has a fixed point—the centre of star $K_{1,3}$ has to be mapped onto itself.

It is interesting to note that all other embeddable graphs of order n and size $n-1$ can be embedded without fixed points. The author proved in [10] the following:

Theorem 3. *Let G be a graph of order n. If $|E(G)| \leqslant n-1$, G is embeddable and $G \neq K_{1,2} \cup K_3$, $G \neq K_{1,3} \cup K_3$ then there is an embedding of G without fixed points.*

These results can be strengthened by specifying the structure of embeddings. In [12] the author proved that every graph of order n and size at most $n-2$ has an embedding which is a cycle of length n. Moreover, graphs of order n and size $n-1$ or n with cyclic embeddings are considered in [13] and [7], respectively.

Much less is known about digraphs. In [1] the authors proved that every digraph of order $n \geqslant 3$ and size at most n is contained in a self-complementary digraph of order n. This result is far from expectation also formulated in [1].

Conjecture 1. Let D be a digraph of order n. If $|A(D)| \leqslant 2n-3$ then D is contained in a self-complementary digraph of order n unless n is even and D is isomorphic to the digraph D' or its converse, where $V(D') = \{v_i; i = 1, \ldots, n\}$, $A(D') = \{(v_1, v_j), (v_j, v_1); j = 2, \ldots, n-2, (v_1, v_{n-1}), (v_1, v_n), (v_{n-1}, v_n)\}$.

The next result associated with this problem is given in [11]. The authors proved that every digraph of order $n \geqslant 2$ and size less than $\frac{3}{2}(n-2)$ is contained in a self-complementary digraph of order n.

It is clear that any digraph contained in a self-complementary digraph of the same order is embeddable. Hence every digraph of order n and size less than $\frac{3}{2}(n-2)$ is embeddable. Our aim is to prove:

Theorem 4. *Let D be a digraph of order n. If $|A(D)| \leqslant \frac{7}{4}n - 81$ then D is embeddable.*

In fact we will prove a bit stronger result. Using this additional property of embeddings we will simplify and shorten the proof.

Theorem 5. *Let D be a digraph of order n. If $|A(D)| \leqslant \frac{7}{4}n - 81$ then there is an embedding σ of D without fixed points, i.e. $\sigma(x) \neq x$ for $x \in V(D)$.*

Observe that, by Theorem 2, every digraph of order n and size at most $n-2$ has an embedding without fixed points.

Sketch of the proof of Theorem 5. Let D be a digraph of order n and size at most $\frac{7}{4}n - 81$. For $W \subset V(D)$ we will denote by $D - W$ the subdigraph of D induced by $V(D) - W$. For $v \in V(D)$ write $d(v) = d^+(v) + d^-(v)$ and set $N(v) = \{s \in V(D) : (v, s) \in A(D) \vee (s, v) \in A(D)\}$. For $W \subset V(D)$ write $N(W) = \bigcup_{v \in W} N(v)$. For brevity we will call *fpf-embedding* (*fixed-point-free embedding*) of D the embedding of D without fixed points.

The proof is by induction on n. If $n \leqslant 105$ then $\frac{7}{4}n - 81 \leqslant n-2$ and by Theorem 2 there is an fpf-embedding of D. Hence let us suppose that $n \geqslant 106$, D is a digraph of order n and size at most $\frac{7}{4}n - 81$ and the theorem is true for digraphs with order less than n. Without loss of generality we may assume that $|A(D)| = \lfloor \frac{7}{4}n - 81 \rfloor$.

The rest of the paper is organized as follows. In the next section we will prove two auxiliary observations and two auxiliary lemmas. Then, in the third section, we will present the main part of the proof. $\square$

2. Some observations and lemmas

Observation 1. *Let $d(v) \leqslant 7$ for every vertex $v \in V(D)$. Then there is an fpf-embedding of D.*

Proof of Observation 1. Observe that there is a vertex $w \in V(D)$ such that $d(w) \geqslant 2$. By induction hypothesis there is an fpf-embedding σ' of $D' = D - w$. We will consider pairs of vertices v_p, v_u such that $\sigma'(v_p) = v_u$. There are $n-1$

such pairs. Observe that if $(\sigma')^{-1}(s) \notin N(v_p)$ and $\sigma'(s) \notin N(v_u)$ for every $s \in N(w)$ then σ defined by $\sigma(v_p) = w$, $\sigma(w) = v_u$, $\sigma(v) = \sigma'(v)$ for $v \in V(D) - \{v_p, w\}$, where $\sigma'(v_p) = v_u$, is an fpf-embedding of D. It is easily seen that

$$|\{v \in V(D) - \{w\} : (\sigma')^{-1}(s) \in N(v) \vee \sigma'(s) \in N(v),$$

where

$$s \in N(w)\}| \leqslant \sum_{s \in N(w)} d((\sigma')^{-1}(s)) + \sum_{s \in N(w)} d(\sigma'(s)) \leqslant 2 \times 7 \times |N(w)| \leqslant 2 \times 7^2.$$

Hence we obtain that for at least $n - 1 - 2 \times 7^2 > 0$ pairs of vertices v_p, v_u such that $\sigma'(v_p) = v_u$, the above defined permutation σ is an fpf-embedding of D. □

Observation 2. *If there is an isolated vertex in D then there is an fpf-embedding of D.*

Proof of Observation 2. Let $u \in V(D)$ such that $d(u) = 0$. Let us first suppose that there is a vertex $v \in V(D)$ such that $d(v) \geqslant 4$. Then, by induction hypothesis, there is an fpf-embedding σ' of $D' = D - \{u, v\}$ and hence $(uv)\sigma'$ is an fpf-embedding of D. If $d(v) \leqslant 3$ for every vertex $v \in V(D)$ then, by Observation 1, there is an fpf-embedding of D. □

Lemma 1. *If there are at least two vertices of degree* 1 *in D then there is an fpf-embedding of D.*

Proof of Lemma 1. Let u_1, u_2 be vertices of D such that $d(u_1) = d(u_2) = 1$ and let z_i be the only neighbour of u_i, $i = 1, 2$. We have divided the proof into several parts.

(I) Let us first suppose that $d(z_1) \geqslant 4$. Then, by induction hypothesis, there is an fpf-embedding σ' of $D' = D - \{u_1, z_1\}$ and hence $(u_1 z_1)\sigma'$ is an fpf-embedding of D.

(II) Let us suppose that $d(z_1) = 1$. If there is a vertex v such that $d(v) \geqslant 5$ then, by induction hypothesis, there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, v\}$ and hence $(u_1 z_1 v)\sigma'$ is an fpf-embedding of D. If $d(v) \leqslant 4$ for every $v \in V(D)$ then, by Observation 1, there is an fpf-embedding of D.

(III) Let us suppose that $|N(z_1)| = 2$. We set $N(z_1) = \{u_1, z_1'\}$. In this case either $d(z_1) = 2$ or $d(z_1) = 3$, $(z_1, z_1'), (z_1', z_1) \in A(D)$. Let us first consider two possibilities:

(III.1.) Let us suppose that there is a vertex $w \in V(D) - \{u_1, z_1, z_1'\}$ such that vertices z_1', w are not joined by the symmetric arc (i.e. $(w, z_1') \notin A(D)$ or $(z_1', w) \notin A(D)$) and vertices z_1', w cover at least 6 arcs. By induction hypothesis, there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', w\}$ and hence at least one of permutations $\sigma_1 = (z_1' z_1 w u_1)\sigma'$, $\sigma_2 = (z_1 z_1' u_1 w)\sigma'$ is an fpf-embedding of D.

(III.2.) Let us suppose that there is a vertex $w \in V(D) - \{u_1, z_1, z_1'\}$ such that $d(w) \geqslant 5$ and $N(z_1') \subset \{z_1, w\}$. Then, by induction hypothesis, there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', w\}$ and hence $(z_1' u_1 z_1 w)\sigma'$ is an fpf-embedding of D.

If $d(z_1') \geqslant 5$ then, by Part I, we may assume that $z_1' \neq z_2$. Taking $w = u_2$ we obtain possibility III.1 with fpf-embeddings of D σ_1 and σ_2. We may assume that $d(z_1') \leqslant 4$. By Observation 1 we may assume that there is at least one vertex of degree at least 8 in D. Let $v \in V(D)$ be a vertex of D such that $d(v) \geqslant 8$. Then $v \neq z_1'$. By possibility III.1, we may assume that $(v, z_1'), (z_1', v) \in A(D)$. By possibility III.2, we may assume that $d(z_1') = 4$ and z_1, z_1' are not joined by the symmetric arc. From this $|N(z_1')| = 3$. It is clear that there is a vertex $v_1 \in V(D) - \{u_1, z_1, z_1'\}$ such that $v_1 \in N(v) - N(z_1')$. By Part I, we may assume that v_1 is not of degree 1. Then $d(v_1) \geqslant 2$, vertices z_1', v_1 cover at least 6 arcs and, by possibility III.1, there is an fpf-embedding of D.

(IV) Let us finally suppose that $|N(z_1)| = 3 = d(z_1)$. We set $N(z_1) = \{u_1, z_1', z_1''\}$. Without loss of generality we may assume that $d(z_1') \leqslant d(z_1'')$. By Parts I–III we may assume that $|N(z_2)| = 3 = d(z_2)$. Let us first consider five possibilities:

(IV.1.) Let us suppose that z_1', z_1'' are joined by the symmetric arc (i.e. $(z_1', z_1''), (z_1'', z_1') \in A(D)$) and let vertices z_1', z_1'', z_2 cover at least 10 arcs. It is clear that since z_2 is not incident with a symmetric arc, $z_1' \neq z_2 \neq z_1''$. Then, by induction hypothesis, there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1'', u_2, z_2\}$. If $N(z_2) \neq \{u_2, z_1', z_1''\}$

then at least one of two permutations $\sigma_1 = (z_2u_1)(z_1'z_1)(z_1''u_2)\sigma'$ or $(z_2u_1)(z_1''z_1)(z_1'u_2)\sigma'$ is an fpf-embedding of D. If $N(z_2) = \{u_2, z_1'z_1''\}$ then at least one of three permutations $(z_1'z_2u_2)(z_1''z_1u_1)\sigma'$ or $(z_1z_1')(z_2z_1'')(u_1u_2)\sigma'$ or σ_1 is an fpf-embedding of D.

(IV.2) Let us suppose that z_1', z_1'' are joined by the symmetric arc, $N(z_1') \subset \{z_1, z_1'', z_2\}$ and there is a vertex $w \in V(D) - \{z_1', z_1''\}$ such that $d(w) \geqslant 4$. Since z_2 is not incident with a symmetric arc, $z_1' \neq z_2 \neq z_1''$. By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', w, u_2, z_2\}$. Since $\sigma'(z_1'') \neq z_1''$,the permutation $(z_1z_1')(u_1w)(u_2z_2)\sigma'$ is an fpf-embedding of D.

(IV.3) Let us suppose that z_1', z_1'' are not joined by the symmetric arc and let vertices z_1', z_1'' cover at least 6 arcs. By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1''\}$. Then at least one of two permutations $(z_1z_1'')(u_1z_1')\sigma'$ or $(z_1z_1')(u_1z_1'')\sigma'$ is an fpf-embedding of D.

(IV.4) Let us suppose that $N(z_1') \subset \{z_1, z_1''\}$ and there is a vertex $w \in V(D) - \{z_1', z_1''\}$ such that $d(w) \geqslant 4$. By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', w\}$. Since $\sigma'(z_1'') \neq z_1''$, the permutation $(z_1z_1')(u_1w)\sigma'$ is an fpf-embedding of D.

(IV.5) Let us suppose that $N(z_1') - \{z_1, z_1''\} = \{s_1\}$, vertices z_1', s_1 are not joined by the symmetric arc, $s_1 \notin N(z_1'')$ and there is a vertex $w \in V(D) - \{u_1, z_1, z_1', z_1'', s_1\}$ such that $w \notin N(z_1'')$ and vertices z_1', z_1'', s_1, w cover at least 10 arcs. By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1'', s_1, w\}$. Then $(wu_1)(s_1z_1')(z_1z_1'')\sigma'$ is an fpf-embedding of D.

Let us first suppose that z_1', z_1'' are joined by the symmetric arc. Since z_2 is not incident with a symmetric arc, $z_1' \neq z_2 \neq z_1''$. Observe that z_1', z_1'', z_2 cover at least 7 arcs. By possibility IV.1 we may assume that z_1', z_1'', z_2 cover at most 9 arcs. Then $d(z_1') \leqslant d(z_1'') \leqslant 6$. By Observation 1 there is a vertex v of D such that $d(v) \geqslant 8$. Then $z_1' \neq v \neq z_1''$. If $N(z_1') \subset \{z_1, z_1'', z_2\}$ taking $w = v$ we obtain possibility IV.2. Hence we can assume that $N(z_1') - \{z_1, z_1'', z_2\} \neq \emptyset$. Similarly we can assume that $N(z_1'') - \{z_1, z_1', z_2\} \neq \emptyset$. It is not difficult to check that there are three cases:

(i) $d(z_1') = 4$, $d(z_1'') = 5$ and $z_2 \notin N(z_1')$, $z_2 \in N(z_1'')$,
(ii) $d(z_1') = d(z_1'') = 5$ and $z_2 \in N(z_1')$, $z_2 \in N(z_1'')$,
(iii) $d(z_1') = d(z_1'') = 4$ and $z_2 \notin N(z_1')$, $z_2 \notin N(z_1'')$.

(i) $d(z_1') = 4$, $d(z_1'') = 5$ and $z_2 \notin N(z_1')$, $z_2 \in N(z_1'')$: Set $N(z_2) - \{u_2, z_1''\} = \{t\}$. It is easy to see that since $t \neq z_1'$, vertices z_1'', t are not joined by the symmetric arc. Observe that vertices z_1'', t cover at least 6 arcs. Applying IV.3 for $D' = D - \{u_2, z_2, z_1'', t\}$, we see that there is an fpf-embedding of D.

(ii) $d(z_1') = d(z_1'') = 5$ and $z_2 \in N(z_1')$, $z_2 \in N(z_1'')$: By Observation 1 there is a vertex v of D such that $d(v) \geqslant 8$. By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1'', u_2, z_2, v\}$. Let us first suppose that $z_1', z_1'' \in N(v)$. Then $(z_1'v)(z_2u_2z_1''u_1z_1)\sigma'$ is an fpf-embedding of D. Let us suppose that $z_1' \notin N(v)$ or $z_1'' \notin N(v)$. Without loss of generality we may assume that z_1' is not a neighbour of v. Then at least one of four permutations $(vz_1z_1''u_2)(z_2z_1'u_1)\sigma'$ or $(vz_1z_1'u_1)(z_1''u_2z_2)\sigma'$or $(vz_1z_2z_1'u_1z_1''u_2)\sigma'$ or $(vz_2z_1''u_1)(z_1z_1'u_2)\sigma'$ is an fpf-embedding of D.

(iii) $d(z_1') = d(z_1'') = 4$ and $z_2 \notin N(z_1')$, $z_2 \notin N(z_1'')$: Set $N(z_1') - \{z_1, z_1''\} = \{s_1\}$. If $d(s_1) \leqslant 6$ then by Observation 1 there is a vertex w of D such that $d(w) \geqslant 8$. If $d(s_1) \geqslant 7$ then let w be a vertex of D such that $w \notin \{u_1, z_1, z_1', z_1'', u_2, z_2, s_1\}$. In both cases vertices z_1', z_1'', s_1, w cover at least 11 arcs. By possibility IV.5 we may assume that either $w \in N(z_1'')$ or $s_1 \in N(z_1'')$. By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1'', s_1, w, u_2, z_2\}$. Let us first suppose that $N(z_1'') = \{z_1, z_1', w\}$. Then $(z_1's_1)(z_1z_1'')(u_1z_2)(wu_2)\sigma'$ is an fpf-embedding of D. Let us suppose that $N(z_1'') = \{z_1, z_1', s_1\}$. Then $(z_1'z_1)(wz_1'')(s_1u_2)(u_1z_2)\sigma'$ is an fpf-embedding of D.

Hence we may assume that vertices z_1', z_1'' are not joined by the symmetric arc. It is clear that vertices z_1', z_1'' cover at least 2 arcs. By possibility IV.3 we may assume that vertices z_1', z_1'' cover at most 5 arcs. Then $d(z_1') \leqslant d(z_1'') \leqslant 4$. By Observation 1 there is a vertex v of D such that $d(v) \geqslant 8$. Then $z_1' \neq v \neq z_1''$. If $N(z_1') \subset \{z_1, z_1''\}$ taking $w = v$ we obtain possibility IV.4. Hence we can assume that $N(z_1') - \{z_1, z_1''\} \neq \emptyset$. It is not difficult to check that there are three cases:

(a) $z_1' \notin N(z_1'')$ and $d(z_1') = d(z_1'') = 2$,

(b) $z_1' \notin N(z_1'')$ and $d(z_1') = 2$, $d(z_1'') = 3$,

(c) $z_1' \in N(z_1'')$ and $d(z_1') = d(z_1'') = 3$.

In every case let us set $N(z_1') - \{z_1, z_1''\} = \{s_1\}$. If $d(s_1) \leqslant 7$ then by Observation 1 there is a vertex w of D such that $d(w) \geqslant 8$. If $d(s_1) \geqslant 8$ then let w be a vertex of D such that $w \notin \{u_1, z_1, z_1', z_1'', u_2, z_2, s_1\}$. If $d(s_1) \geqslant 8$ and vertices s_1, w are neighbours, by Part I, we may assume that $d(w) \geqslant 2$. It is clear that vertices z_1', z_1'', s_1, w cover at least 11 arcs. By possibility IV.5 we may assume that $N(z_1'') \cap \{w, s_1\} \neq \emptyset$.

(a) $z_1' \notin N(z_1'')$ and $d(z_1') = d(z_1'') = 2$: By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1'', s_1, w\}$. Then $(s_1 z_1')(w z_1'')(z_1 u_1)\sigma'$ is an fpf-embedding of D.

(b) $z_1' \notin N(z_1'')$ and $d(z_1') = 2$, $d(z_1'') = 3$: By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1'', s_1, w\}$. If either $N(z_1'') = \{z_1, s_1, w\}$ or $N(z_1'') = \{z_1, s_1\}$ then $(s_1 z_1')(w z_1'')(z_1 u_1)\sigma'$ is an fpf-embedding of D. If $N(z_1'') = \{z_1, w\}$ then $(u_1 z_1)(z_1'' s_1 z_1' w)\sigma'$ is an fpf-embedding of D. Hence we may assume that $N(z_1'') \cap \{s_1, w\} \neq \emptyset$ and $N(z_1'') \not\subset \{z_1, s_1, w\}$. Set $\{s_2\} = N(z_1'') - \{z_1, s_1, w\}$. By induction hypothesis there is an fpf-embedding σ'' of $D'' = D - \{u_1, z_1, z_1', z_1'', s_1, w, s_2\}$. Then $(z_1' s_1)(z_1'' w)(z_1 u_1 s_2)\sigma''$ is an fpf-embedding of D.

(c) $z_1' \in N(z_1'')$ and $d(z_1') = d(z_1'') = 3$: Let us recall that $w \notin \{u_2, z_2\}$. Observe that since $w \neq u_2$, we have $z_2 \neq z_1''$. Let us first suppose that $z_2 \notin \{z_1', s_1\}$. By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1'', s_1, w, u_2, z_2\}$. If $N(z_1'') = \{z_1, z_1', w\}$ then $(z_1' s_1)(z_1 z_1'')(u_1 z_2)(w u_2)\sigma'$ is an fpf-embedding of D. If $N(z_1'') = \{z_1, z_1', s_1\}$ then $(z_1' z_1)(w z_1'')(s_1 u_2)(u_1 z_2)\sigma'$ is an fpf-embedding of D. Let us suppose that $z_2 = z_1'$. It is clear that in this case $z_1'' \notin N(s_1)$ and $s_1 = u_2$. Hence vertices s_1, w are not neighbours. By induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1'', s_1, w\}$ and then $(u_1 z_1)(z_1' s_1)(z_1'' w)\sigma'$ is an fpf-embedding of D. We may assume that $z_2 = s_1$. Let us set $N(z_2) = \{u_2, z_2', z_2''\}$. By the above arguments in Part IV we may assume that $d(z_2') = d(z_2'') = 3$, $z_2' \in N(z_2'')$. It is not difficult to see that then $\{z_1', z_1''\} = \{z_2', z_2''\}$. Hence, since $w \neq z_2$, vertices s_1, w are not neighbours. Thus by induction hypothesis there is an fpf-embedding σ' of $D' = D - \{u_1, z_1, z_1', z_1'', s_1, w\}$ and then $(u_1 z_1)(z_1' s_1)(z_1'' w)\sigma'$ is an fpf-embedding of D.

Let us show that the proof is correct for $z_1 = z_2$, too. By Part I and by possibility IV.4 we may assume that $d(z_1) = 2$. By Observation 1 we may assume that there is at least one vertex of degree at least 8 in D. Let $v \in V(D)$ be a vertex of D such that $d(v) \geqslant 8$. Then $v \neq z_1' = u_2$ and by possibility III.1 (or III.2) there is an fpf-embedding of D. □

In the proof of the next lemma we will use Hall's theorem [8,5].

A set M of independent edges in a graph G is called a *matching*. A set M is a matching of $U \subset V(G)$ if every vertex of U is incident with an edge in M.

Theorem 6 (*P. Hall*). *Let G be a bipartite graph with sets A, B of bipartition of $V(G)$. Then G contains a matching of A if and only if $|N(S)| \geqslant |S|$ for all $S \subset A$.*

Lemma 2. *Given $k \geqslant 2$ suppose that there are at least $2(k+1)$ vertices $v_1, \ldots, v_{2k+2}$ such that $d(v_i) \leqslant k$, $i = 1, \ldots, 2k+2$, $N(v_i) \cap N(v_j) = \emptyset$ for $i \neq j$, $i, j \in \{1, \ldots, 2k+2\}$ and $\sum_{i=1}^{2k+2} d(v_i) \geqslant 4(k+1)$. Then there is an fpf-embedding of D.*

Proof of Lemma 2. Since $\sum_{i=1}^{2k+2} d(v_i) \geqslant 4(k+1)$, by induction hypothesis, there is an fpf-embedding σ' of $D' = D - \{v_i; i = 1, \ldots, 2k+2\}$.

For $v \in V(D) - \{v_1, \ldots, v_{2k+2}\}$ we fix $\sigma(v) = \sigma'(v)$. To set $\sigma(v_i)$, $i = 1, \ldots, 2k+2$, let us consider the bipartite graph G with sets A, B of bipartition of $V(G)$ such that $A = \{v_1, \ldots, v_{2k+2}\} \times \{0\}$, $B = \{v_1, \ldots, v_{2k+2}\} \times \{1\}$. For $i = 1, \ldots, 2k+2$ vertices $(v_i, 0)$, $(v_i, 1)$ are not joined by an edge in G. For $i \neq j$, $i, j \in \{1, \ldots, 2k+2\}$ vertices $(v_i, 0)$, $(v_j, 1)$ are joined by an edge in G if and only if $\sigma'(N(v_j)) \cap N(v_j) = \emptyset$. Observe that if vertices $(v_i, 0)$, $(v_j, 1)$, $i, j \in \{1, \ldots, 2k+2\}$ are joined by an edge in G then we can put $\sigma(v_i)$ as v_j.

For $i = 1, \ldots, 2k+2$, since $d(v_i) \leqslant k$, we obtain $d((v_i, 0)) \geqslant 2k+2-(k+1) = k+1$ and $d((v_j, 1)) \geqslant 2k+2-(k+1) = k+1$. It is obvious that for $S \subset A$, $|S| \leqslant k+1$ we have $|N(S)| \geqslant k+1$ and then $|N(S)| \geqslant |S|$.

Observe that if $S \subset A$ and $|S| > k+1$ then $N(S) = B$ and hence $|N(S)| \geqslant |S|$. On the contrary, suppose that there is a vertex $(v_l, 1) \in B - N(S)$, $l \in \{1, \ldots, 2k+2\}$. Then $N((v_l, 1)) \subset A - S$. Hence $k+1 \leqslant |N((v_l, 1))| \leqslant |A - S| = 2k+2-|S| < k+1$, a contradiction. By Hall's theorem there is the matching M of A. Thus for every $i \in \{1, \ldots, 2k+2\}$ we set $\sigma(v_i) = v_j$, $j \in \{1, \ldots, 2k+2\}$ where $(v_i, 0)$, $(v_j, 1)$ are incident with the same edge in M. □

3. Proof of Theorem 5—the main part

By Observations 2, 1 we may assume that $d(v)\geqslant 2$ for every vertex v in D except for at most one of degree 1. By Lemma 2 we may assume that there are at most 7 vertices $v_1,\ldots,v_7$ such that $2\leqslant d(v_i)\leqslant 3$, $i=1,\ldots,7$, $N(v_i)\cap N(v_j)=\emptyset$ for $i\neq j$, $i,j\in\{1,\ldots,7\}$. The proof falls into several parts.

Part A: We choose the set S with maximum cardinality among all sets of vertices of degree 2 or 3 with pairwise disjoined sets of neighbours. Then $|S|\leqslant 7$ and hence $|N(S)|\leqslant 21$. It is clear that every vertex of degree 2 or 3 is a neighbour of at least one vertex from $N(S)$. Write $V_2=\{v\in V(D)-N(S): d(v)=|N(v)|=2\}$, $V_z=\{v\in V(D)-N(S): d(v)=2, |N(v)|=1\}$, $V_3=\{v\in V(D)-N(S): d(v)=3\}$. Since $\sum_{v\in N(S)} d(v)\geqslant |V_2|+2|V_z|+|V_3|$ and there is at least one vertex of degree 1, we obtain that

$$\begin{aligned}\frac{7}{2}n-162 &\geqslant \sum_{v\in V(D)} d(v)\\ &\geqslant 2|V_2|+2|V_z|+3|V_3|+(|V_2|+2|V_z|+|V_3|)+4(n-1-|V_2|-|V_z|-|V_3|-|N(S)|)+1\times 1.\end{aligned}$$

Hence $|V_2|\geqslant n/2+159-4|N(S)|\geqslant n/2+75$.

Part B: We will denote by W_2 the subset of V_2 such that for every $v\in W_2$ holds $N(v)\cap V_2=\emptyset$. Since at least one neighbour of every vertex of degree 2 is in $N(S)$, the set V_2-W_2 contains vertices with exactly one neighbour in $N(S)$ and exactly one neighbour in V_2. Let us consider vertices from V_2-W_2:

Case B.1: There are vertices $v_1, v_2\in V_2-W_2$ such that $v_1\in N(v_2)$ and there is a vertex x of D such that $\{x\}=N(v_1)\cap N(v_2)\cap N(S)$. By Observation 1 we may assume that there is a vertex $w\in V(D)$ such that $d(w)\geqslant 8$. Let us first suppose that vertices x, w are not joined by the symmetric arc. By induction hypothesis there is an fpf-embedding σ' of $D'=D-\{v_1,v_2,x,w\}$ and then at least one of permutations $(xv_1)(wv_2)\sigma'$, $(xv_2)(wv_1)\sigma'$ is an fpf-embedding of D. Hence we may assume that $(x,w),(w,x)\in A(D)$. Then $d(x)\geqslant 4$. Let v_3 be a vertex of $V_2-\{v_1,v_2\}$. Then vertices x, v_3 are not joined by the symmetric arc and x, v_3 cover at least 6 arcs. By induction hypothesis there is an fpf-embedding σ' of $D'=D-\{v_1,v_2,x,v_3\}$ and then at least one of permutations $(xv_1)(v_3v_2)\sigma'$, $(xv_2)(v_3v_1)\sigma'$ is an embedding of D.

Case B.2: There are vertices $v_1, v_2, u_1\ u_2\in V_2-W_2$ such that $v_1\in N(v_2)$, $u_1\in N(u_2)$ and there are vertices $x_1, x_2\in N(S)$, $x_1\neq x_2$, $x_1\in N(v_1)\cap N(u_1)$, $x_2\in N(v_2)\cap N(u_2)$. If vertices x_1, x_2 cover at least 9 arcs then by induction hypothesis there is an fpf-embedding σ' of $D'=D-\{v_1,v_2,u_1,u_2,x_1,x_2\}$ and at least one of permutations $(v_1x_1)(u_1x_2)(v_2u_2)\sigma'$, $(x_1v_2)(x_2u_2)(v_1u_1)\sigma'$, $(x_1v_1)(x_2u_2)(u_1v_2)\sigma'$ is an fpf-embedding of D. Hence we can assume that vertices x_1, x_2 cover at most 8 arcs. Especially, $d(x_i)\leqslant 6$ for $i=1,2$. By Observation 1 we may assume that there is a vertex $w\in V(D)-\{x_1,x_2\}$ such that $d(w)\geqslant 8$. By induction hypothesis there is an fpf-embedding σ' of $D'=D-\{v_1,v_2,u_1,u_2,x_1,w\}$. Since $\sigma'(x_2)\neq x_2$, $(v_2u_2)(wu_1)(v_1x_1)\sigma'$ is an fpf-embedding of D.

Case B.3: There are vertices $v_1, v_2, u_1, u_2\in V_2-W_2$ such that $v_1\in N(v_2)$, $u_1\in N(u_2)$ and there are vertices $x_1, x_2, y_1\in N(S)$ such that $x_1\in N(v_1)$, $y_1\in N(u_1)$, $x_2\in N(v_2)\cap N(u_2)$ and $x_1\neq y_1\neq x_2\neq x_1$. If vertices x_1, y_1 cover at least 7 arcs then by induction hypothesis there is an fpf-embedding σ' of $D'=D-\{v_1,v_2,u_1,u_2,x_1,y_1\}$ and, since $\sigma'(x_2)\neq x_2$, $(v_2u_2)(x_1v_1)(u_1y_1)\sigma'$ is an fpf-embedding of D. Hence we can assume that vertices x_1, y_1 cover at most 6 arcs. Especially, $d(x_1)\leqslant 5$ and $d(y_1)\leqslant 5$. Let us first suppose that $d(x_2)\geqslant 7$. Then by induction hypothesis there is an fpf-embedding σ' of $D'=D-\{v_1,v_2,u_1,u_2,x_1,x_2\}$ and $(x_1v_2)(x_2u_2)(u_1v_1)\sigma'$ is an fpf-embedding of D. Hence we can assume that $d(x_2)\leqslant 6$. By Observation 1 there is a vertex $w\in V(D)-\{x_1,x_2,y_1\}$ such that $d(w)\geqslant 8$. By induction hypothesis there is an fpf-embedding σ' of $D'=D-\{v_1,v_2,u_1,u_2,x_1,x_2,y_1,w\}$. If vertices x_1, y_1 are not joined by the symmetric arc and vertices x_2, w are not joined by the symmetric arc then at least one of permutations $(x_1v_1)(y_1v_2)(x_2u_2)(wu_1)\sigma'$ or $(x_1v_2)(y_1v_1)(x_2u_2)(wu_1)\sigma'$ or $(x_1v_1)(y_1v_2)(x_2u_2)(wu_1)\sigma'$ or $(x_1v_2)(y_1v_1)(x_2u_1)(wu_2)\sigma'$ is an fpf-embedding of D. If the vertices x_1, x_2 are not neighbours and the vertices w, y_1 are not joined by the symmetric arc then at least one of permutations $(x_1v_1)(x_2v_2)(y_1u_1)(wu_2)\sigma'$ or $(x_1v_1)(x_2v_2)(y_1u_2)(wu_1)\sigma'$ is an fpf-embedding of D. Analogously, if vertices y_1, x_2 are not neighbours and vertices w, x_1 are not joined by the symmetric arc then there is an fpf-embedding of D. If vertices w, x_1 are not neighbours then at least one of permutations $(x_1u_2)(x_2u_1)(y_1v_1)(wv_2)\sigma'$, $(x_1u_1)(x_2u_2)(y_1v_2)(wv_1)\sigma'$ is an fpf-embedding of D. Analogously, if vertices w, y_1 are not neighbours then there is an fpf-embedding of D. Hence we may assume that every case (i), (ii), (iii) and (iv) holds:

(i) vertices x_1, y_1 are joined by the symmetric arc or vertices x_2, w are joined by the symmetric arc,
(ii) vertices x_1, x_2 are neighbours or vertices w, y_1 are joined by the symmetric arc,
(iii) vertices y_1, x_2 are neighbours or vertices w, x_1 are joined by the symmetric arc,
(iv) vertices w, x_1 are neighbours and vertices w, y_1 are neighbours.

It is not difficult to check that then at least one of permutations $(wy_1)(x_1v_2)(x_2u_2)(u_1v_1)\sigma'$, $(wx_1)(y_1u_2)(x_2v_2)(v_1u_1)\sigma'$, $(wv_1u_1v_2x_1x_2u_2y_1)\sigma'$ is an fpf-embedding of D.

Part C: Let us recall that $V_2 - W_2$ is the set of vertices from $V(D) - N(S)$ with exactly one neighbour in $N(S)$ and exactly one neighbour in $V_2 - W_2$. It is clear that vertices of $V_2 - W_2$ create pairs of vertices v, u such that u is the only neighbour of v in $V_2 - W_2$ and v is the only neighbour of u in $V_2 - W_2$. We choose one vertex from each such pair of vertices from $V_2 - W_2$. Let U_2 be the set of chosen vertices. Therefore $|V_2 - W_2| = 2|U_2|$. Since $|N(S)| \leqslant 21$, by Cases B.1, B.2 and B.3 in Part B, we obtain that $|U_2| \leqslant 10$. Since $|V_2| \geqslant n/2 + 75$, we obtain $|W_2 \cup U_2| \geqslant n/2 + 65$.

Part D: Let us consider the bipartite subdigraph B_0 of the digraph D such that sets of vertices $W_2 \cup U_2$ and $V(D) - (W_2 \cup U_2)$ are two classes of bipartition of vertices of B_0 and every arc of D adjacent to exactly one vertex from $W_2 \cup U_2$ and exactly one vertex from $V(D) - (W_2 \cup U_2)$ is an arc of B_0. It is clear that none of the two vertices from $W_2 \cup U_2$ are neighbours in D. Then every arc adjacent to any vertex of $W_2 \cup U_2$ is an arc of B_0. Every vertex from $W_2 \cup U_2$ is of degree 2. Therefore the subdigraph B_0 has at least $2|W_2 \cup U_2| \geqslant n + 130$ arcs. Since $|V(B_0)| = |V(D)| = n$, there is a cycle C_0 in the subdigraph B_0. Let C_0 be the cycle of length $2k_0$, $k_0 \geqslant 2$. Set $V(C_0) = \{x_i^0, y_i^0; i = 1, \ldots, k_0\}$ such that $x_i^0 \in W_2 \cup U_2$ for $i = 1, \ldots, k_0$. Since every vertex from $W_2 \cup U_2$ has at least one neighbour from $N(S)$, we have $|N(S) \cap V(C_0)| \geqslant \lceil k_0/2 \rceil \geqslant 1$. Since $|N(S)| \leqslant 21$, $k_0 \leqslant 42$. Let us first suppose that vertices x_i^0, y_i^0, $i = 1, \ldots, k_0$ cover at least $\lceil \frac{7}{2}k_0 \rceil$ arcs in D. Then by induction hypothesis there is an fpf-embedding σ' of $D' = D - V(C_0)$. If $k_0 > 2$ then $(x_1^0 y_1^0) \cdots (x_{k_0}^0 y_{k_0}^0)\sigma'$ is an fpf-embedding of D. Let $k_0 = 2$. Then either at least one of two permutations $\sigma_1 = (x_1^0 y_1^0)(x_2^0 y_2^0)\sigma'$, $\sigma_2 = (x_1^0 y_2^0)(x_2^0 y_1^0)\sigma'$ is an fpf-embedding of D or one of four possibilities holds:

(i) $d^+(x_1^0) = d^-(x_2^0) = 2$,
(ii) $d^-(x_1^0) = d^+(x_2^0) = 2$,
(iii) $(x_i^0, y_1^0), (y_2^0, x_i^0) \in A(D)$ for $i = 1, 2$,
(iv) $(y_1^0, x_i^0), (x_i^0, y_2^0) \in A(D)$ for $i = 1, 2$.

We can assume that neither σ_1 nor σ_2 is an fpf-embedding of D. By induction hypothesis there is an fpf-embedding σ'' of $D'' = D - \{x_1^0, x_2^0\}$. Since $\sigma''(y_i^0) \neq y_i^0$, $i = 1, 2$, in every case (i)–(iv), $(x_1^0 x_2^0)\sigma''$ is an fpf-embedding of D. Now we may assume that vertices x_i^0, y_i^0, $i = 1, \ldots, k_0$ cover less then $\lceil \frac{7}{2}k_0 \rceil$ arcs in D.

Part E: Let us consider the bipartite digraph $B_1 = B_0 - V(C_0) - [(W_2 \cup U_2) \cap N(C_0)]$. It is clear that sets of vertices $(W_2 \cup U_2) - N(C_0)$ and $V(D) - (W_2 \cup U_2) - V(C_0)$ are two classes of bipartition of vertices of B_1. Since none of the two vertices from $W_2 \cup U_2$ are neighbours in D, every arc adjacent to any vertex of $(W_2 \cup U_2) - N(C_0)$ is an arc of B_1. Every vertex from $(W_2 \cup U_2) - N(C_0)$ is of degree 2 in B_1. Therefore the subdigraph B_1 has at least $2|(W_2 \cup U_2) - N(C_0)|$ arcs. Observe that

$$
\begin{aligned}
&|(W_2 \cup U_2) - N(C_0)| \\
&\quad = |W_2 \cup U_2| - |(W_2 \cup U_2) \cap V(C_0)| - |(W_2 \cup U_2) \cap (N(C_0) - V(C_0))| \\
&\quad \geqslant \frac{n}{2} + 65 - k_0 - |(W_2 \cup U_2) \cap (N(C_0) - V(C_0))|
\end{aligned}
$$

and

$$|V(B_1)| = n - 2k_0 - |(W_2 \cup U_2) \cap (N(C_0) - V(C_0))|$$

and

$$|(W_2 \cup U_2) \cap (N(C_0) - V(C_0))| \leqslant \left\lceil \tfrac{3}{2}k_0 \right\rceil - 1.$$

Since $k_0 \leqslant 42$, we have $|V(B_1)| \leqslant |A(B_1)|$ and then there is the cycle C_1 in the subdigraph B_1. Let C_1 be a cycle of length $2k_1$, $k_1 \geqslant 2$. Set $V(C_1) = \{x_i^1, y_i^1 : i = 1, \ldots, k_1\}$ such that $x_i^1 \in (W_2 \cup U_2) - N(C_0)$ for $i = 1,\ldots,k_1$. It is clear that $|N(S) \cap V(C_1)| \geqslant \lceil k_1/2 \rceil \geqslant 1$ and $k_1 \leqslant 42 - 2\lceil k_0/2 \rceil$. If vertices from $V(C_1)$ cover at least $\lceil \frac{7}{2} k_1 \rceil$ arcs in D, then, by induction hypothesis, there is an fpf-embedding of $D' = D - V(C_1)$ and, by Part D, there is an fpf-embedding of D. Hence we may assume that vertices of $V(C_1)$ cover at most $\lceil \frac{7}{2} k_1 \rceil - 1$ arcs in D. Then we consider $B_2 = B_1 - V(C_1) - [(W_2 \cup U_2) \cap N(C_1)] = B_0 - V(C_0) - V(C_1) - [(W_2 \cup U_2) \cap (N(C_0) \cup N(C_1))]$.

Let us first prove that we are able to repeat the above procedure. Let us suppose that $C_0,\ldots,C_p$ be the obtained cycles in D. For $j = 0,\ldots,p$ set $V(C_j) = \{x_i^j, y_i^j; i = 1, \ldots, k_j\}$ such that $x_i^j \in W_2 \cup U_2$ for $i = 1,\ldots,k_j$. It is clear that $|N(S) \cap V(C_j)| \geqslant \lceil k_j/2 \rceil \geqslant 1$ and $\sum_{j=0}^{p} \lceil k_j/2 \rceil \leqslant 21$. We assume that vertices from $V(C_j)$ cover at most $\lceil \frac{7}{2} k_j \rceil - 1$ arcs in D, $j = 0, \ldots, p$. Let us consider the bipartite digraph

$$B_{p+1} = B_0 - \bigcup_{j=0}^{p} V(C_j) - \left[(W_2 \cup U_2) \cap \bigcup_{j=0}^{p} N(C_j) \right].$$

It is clear that sets of vertices $(W_2 \cup U_2) - \bigcup_{j=0}^{p} N(C_j)$ and $V(D) - (W_2 \cup U_2) - \bigcup_{j=0}^{p} V(C_j)$ are two classes of bipartition of vertices of B_{p+1}. Since none of the two vertices from $W_2 \cup U_2$ are neighbours in D, every arc adjacent to any vertex of $(W_2 \cup U_2) - \bigcup_{j=0}^{p} N(C_j)$ is an arc of B_{p+1}. It is not difficult to observe that every vertex of $(W_2 \cup U_2) - \bigcup_{j=0}^{p} N(C_j)$ is of degree 2 in B_{p+1} and hence the subdigraph B_{p+1} has at least $2|(W_2 \cup U_2) - \bigcup_{j=0}^{p} N(C_j)|$ arcs. Observe that

$$\begin{aligned}
&\left| (W_2 \cup U_2) - \bigcup_{j=0}^{p} N(C_j) \right| \\
&\quad = |W_2 \cup U_2| - \left| (W_2 \cup U_2) \cap \bigcup_{j=0}^{p} V(C_j) \right| - \left| (W_2 \cup U_2) \cap \left(\bigcup_{j=0}^{p} N(C_j) - \bigcup_{j=0}^{p} V(C_j) \right) \right| \\
&\quad \geqslant \frac{n}{2} + 65 - \sum_{j=0}^{p} k_j - \left| (W_2 \cup U_2) \cap \left(\bigcup_{j=0}^{p} N(C_j) - \bigcup_{j=0}^{p} V(C_j) \right) \right|
\end{aligned}$$

and

$$|V(B_{p+1})| = n - \sum_{j=0}^{p} 2k_j - \left| (W_2 \cup U_2) \cap \left(\bigcup_{j=0}^{p} N(C_j) - \bigcup_{j=0}^{p} V(C_j) \right) \right|$$

and

$$\left| (W_2 \cup U_2) \cap \left(\bigcup_{j=0}^{p} N(C_j) - \bigcup_{j=0}^{p} V(C_j) \right) \right| \leqslant \sum_{j=0}^{p} \left(\left\lceil \frac{3}{2} k_j \right\rceil - 1 \right).$$

Since $\sum_{j=0}^{p} \lceil k_j/2 \rceil \leqslant 21$, we have $|V(B_{p+1})| \leqslant |A(B_{p+1})|$ and then there is the cycle C_{p+1} in the subdigraph B_{p+1}. Let C_{p+1} be a cycle of length $2k_{p+1}$, $k_{p+1} \geqslant 2$. Set $V(C_{p+1}) = \{x_i^{p+1}, y_i^{p+1} : i = 1, \ldots, k_{p+1}\}$ such that $x_i^{p+1} \in W_2 \cup U_2$. It is clear that $|N(S) \cap V(C_{p+1})| \geqslant \lceil k_{p+1}/2 \rceil \geqslant 1$ and $k_{p+1} \leqslant 42 - 2(\sum_{j=0}^{p} \lceil k_j/2 \rceil)$. If vertices from $V(C_{p+1})$ cover at least $\lceil \frac{7}{2} k_{p+1} \rceil$ arcs in D, then, by induction hypothesis, there is an fpf-embedding of $D' = D - V(C_{p+1})$ and hence, by Part D, we can extend it to an fpf-embedding of D. Hence we may assume that vertices of $V(C_{p+1})$ cover at most $\lceil \frac{7}{2} k_{p+1} \rceil - 1$ arcs in D. Therefore we can repeat the procedure obtaining cycles in D.

Part F: We can assume that $C_0,\ldots,C_p$ are obtained cycles of length $2k_j$, $j = 0, \ldots, p$, respectively. We can assume that vertices from $V(C_j)$ cover at most $\lceil \frac{7}{2} k_j \rceil - 1$ arcs in D. Observe that $V(C_j) \cap V(C_i) = \emptyset$ for $i \neq j$, $i, j \in \{0, \ldots, p\}$ and $|N(S) \cap V(C_j)| \geqslant \lceil k_j/2 \rceil \geqslant 1$ for $j = 0, \ldots, p$. Since $|N(S)| \leqslant 21$, we have $p \leqslant 21$. If $N(S) - \bigcup_{j=0}^{p} V(C_j) \neq \emptyset$ then we are able to repeat the method described in Part E. We can assume that $N(S) \subset \bigcup_{j=0}^{p} V(C_j)$. Hence

$\sum_{v\in N(S)} d(v) \leqslant \sum_{j=0}^{p}(\lceil \frac{7}{2}k_j\rceil - 1) = \sum_{j=0}^{p}\lceil \frac{7}{2}k_j\rceil - (p+1)$. Since every vertex of degree 2 has at least one neighbour in $N(S)$, we obtain that $(W_2 \cup U_2) - \bigcup_{j=0}^{p} N(C_j) = \emptyset$. Let us recall that $\sum_{v\in N(S)} d(v) \geqslant |V_2| \geqslant n/2 + 75$. Observe that

$$21 \geqslant |N(S)| = \left| N(S) \cap \bigcup_{j=0}^{p} V(C_j) \right| = \sum_{j=0}^{p} |N(S) \cap V(C_j)| \geqslant \sum_{j=0}^{p} \left\lceil \frac{k_j}{2} \right\rceil .$$

Since $n \geqslant n/2 + 75$, $p \geqslant 0$, we obtain

$$7 \times 21 - 1 \geqslant \sum_{j=0}^{p} \left\lceil \frac{7}{2} k_j \right\rceil - (p+1) \geqslant \frac{n}{2} + 75 \geqslant 2 \times 75$$

and hence $73 \geqslant 75$, a contradiction. Therefore there is the cycle C_j, $j \in \{0, \ldots, p\}$ such that vertices from $V(C_j)$ cover at least $\lceil \frac{7}{2}k_j \rceil$ arcs in D, by induction hypothesis there is an fpf-embedding of $D' = D - V(C_j)$ and hence, by Part E, there is an fpf-embedding of D. $\square$

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